Self-organized criticality and universality in a nonconservative earthquake model

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We make an extensive numerical study of a two-dimensional nonconservative model proposed by Olami, Feder, and Christensen to describe earthquake behavior [Phys. Rev. Lett. **68**, 1244 (1992)]. By analyzing the distribution of earthquake sizes using a multiscaling method, we find evidence that the model is critical, with no characteristic length scale other than the system size, in agreement with previous results. However, in contrast to previous claims, we find a convergence to universal behavior as the system size increases, over a range of values of the dissipation parameter α. We also find that both ''free'' and ''open'' boundary conditions tend to the same result. Our analysis indicates that, as L increases, the behavior slowly converges toward a power law distribution of earthquake sizes $P(s) \sim s^{-\tau}$ with an exponent $\tau \simeq 1.8$. The universal value of τ we find numerically agrees quantitatively with the empirical value ($\tau = B + 1$) associated with the Gutenberg-Richter law.

I. INTRODUCTION

Earthquakes may be the most dramatic example of self-organized criticality (SOC) [1,2] that can be observed by humans on earth. Most of the time the crust of the earth is at rest, or quiescent. These periods of stasis are punctuated by sudden, thus far unpredictable, bursts, or earthquakes. According to the empirical Gutenberg-Richter (GR) law [3], the distribution of earthquake events is scale free over many orders of magnitude in energy. The GR scaling extends from the smallest measurable earthquakes, which are equivalent to a truck passing by, to the most disastrous that have been recorded, where hundreds of thousands of people have perished.

The relevance of SOC to earthquakes was first pointed out by Bak and Tang [4], Sornette and Sornette [5], and Ito and Matsusaki [6]. According to this theory, plate tectonics provides energy input at a slow time scale into a spatially extended, dissipative system that can exhibit breakdown events via a chain reaction process of propagating instabilities in space and time. The GR law arises from the system of driven plates building up to a critical state with avalanches of all sizes. These above-mentioned authors used a spatially extended (but conservative) cellular automata model as a prototype resembling earthquake dynamics, which gave a power law distribution of avalanches, or earthquakes. This was followed by a study using block-spring models [7], by Carlson and Langer [8], who found characteristic earthquake sizes, rather than asymptotic critical behavior. Later studies of a continuous ''train'' block-spring model by de Sousa Vieira [9] recovered criticality. The train model describes a driven elastic chain sliding over a surface with friction. It was conjectured to be in the same universality class as interface depinning and a model of avalanches in granular piles, which agrees with numerical simulation results [10].

Several groups made lattice representations of the block-spring model. These models were nonconservative [11,12] and were driven uniformly, but did not display SOC. Then Olami, Feder, and Christensen (OFC) introduced a noncon-servative model on a lattice that displayed SOC [13]. In this simplified earthquake model, sites on a lattice are continuously loaded with a force. After a threshold force is reached, the sites transfer part of their force to their local neighborhood when discharging. Each discharge event is accompanied by a local loss in accumulated force from the system, when the force on each element is reset to zero. A uniform driving force is slowly applied to all the elements, and the model is completely deterministic. This conceptually simple and seemingly numerically tractable model reproduces some of the qualitative phenomenology of the statistics of earthquake events such as power law behavior over a range of sizes, intermittency or clustering of large events [13], and lack of predictability [14].

Although SOC and this type of modeling approach have been more or less accepted as reasonable descriptions of the phenomena of earthquakes (see, for example, Ref. [15], and references therein), the OFC model itself has had a controversial existence, both on the theoretical [16–19] and numerical sides [13,20–25]. Initial numerical studies found that the distribution of earthquake sizes obeyed finite size scaling (FSS) over the range of system sizes that could be studied at the time [13]. This placed the nonconservative model into the framework of standard critical behavior. However, these simulations also indicated that there was no universality. In particular the exponents characterizing the power law distributions appeared to vary with both the dissipation parameter α and the form of the boundary condition. If this were the case then the OFC model would be very different from familiar critical systems where most microscopic details are irrelevant and have no effect on critical coefficients. In fact an argument was made [13] that one should not expect universal behavior in far from equilibrium critical phenomena. If this were correct, it would drastically limit the application of any known theoretical tools to these problems.

Another strange aspect was that the dimension D characterizing the scaling of the cutoff in the earthquake size distribution was found numerically to be larger than 2. This is inconsistent with the fact that each site can only discharge a

finite number of times in an earthquake event, requiring $D \leq 2$ for the two-dimensional nonconservative model [16]. This last result, together with the strange lack of universality, cast some doubt on whether the OFC model was actually critical or just close to being critical, with some large, as yet undetermined, length scale beyond which the earthquake distribution would always be cut off. Hwa and Kardar as well as Grinstein *et al.* postulated that conservation of the quantity being transported was required for criticality [26], but the theoretical arguments made do not take into account SOC phenomena such as avalanches and long-term memory associated with the self-organization process (for more details, see Ref. [27]). The fact that the random neighbor version of the nonconservative OFC model is never critical, but has an essential singularity as the conservative limit is approached [18,19], has added to the mystery surrounding the behavior of the lattice model.

In a previous large scale numerical simulation study of the model discussed here, Grassberger [22] also claimed that the model was critical, but found that some of the conclusions of OFC ''have to be modified considerably.'' He argued that the integrated probability distribution of earthquake sizes does not show ordinary FSS over the larger range of systems he was able to study, and ''conjecture(d) that the cutoff of $P_{int}(s)$ becomes a step function for $L \rightarrow \infty$,'' although he did not present direct numerical evidence of this. Note that here we study the ordinary probability distribution of earthquake sizes and not the integrated distribution.

There are a number of important, unresolved questions about the behavior of the model, which have enormous implication for any type of eventual theoretical understanding. Is the nonconservative model on a lattice (or for fixed connectivity matrix) critical? If so, is the critical behavior of the model universal over a range of values of α, or for different boundary conditions? Is it described by a power-law distribution at all? Are there any other universal quantities? What type of data analysis technique besides FSS would be useful to extract the large scale behavior of the nonconservative model? Our numerical study and analysis will address those issues and answer those questions.

Summary

In the first section we review the definition of the OFC model and present some numerical data for the distribution of earthquake sizes using standard FSS. For the range of lattice sizes we have simulated (up to linear size $L=512$), our data confirm the lack of apparent FSS in the model, particularly in the cutoff region. In Sec. III, we present an extensive set of results using a multiscaling method. We analyze the rescaled probability distribution, $\log P(s)/\log L$ [28], in terms of the quantity $D_{av} \equiv \log s/\log L$, with s being the size of an earthquake. We observe that there are no avalanches with D_{av} larger than 2, consistent with the bound imposed on the cutoff s_{co} (see the previous discussion). By analyzing how this distribution behaves for different values of the nonconservation parameter α and system size L, we show how the multiscaled probability distribution tends to

converge to a universal curve as L increases. The direction of convergence on increasing L changes as α varies, enabling us to put fairly firm limits on the asymptotic curve. The model appears not be to described at all by FSS. However, for $s < s_{co}$, the distribution converges toward a power law with a universal exponent $\tau \simeq 1.8$ over a range of α values. Moreover the cutoff in this distribution becomes very sharp as L increases, and its behavior indicates that $s_{co} \rightarrow \text{const}(\alpha)L^2$ as $L \rightarrow \infty$. In Sec. IV we summarize our main conclusions.

II. DEFINITION OF THE MODEL

We consider a two-dimensional square lattice of $L \times L$ sites. At each site i, a force F_i is assigned to be a real variable. Initially, the force at each site is chosen randomly from a uniform distribution between 0 and 1. The dynamics proceeds by two steps in the limit of infinite time scale separation between the slow drive, representing motion of the tectonic plates, and the earthquake process [13].

(1) Increase the force at all sites: Find the largest force F_{max} in the system and increase the force at all sites by the same amount $1 - F_{max}$.

(2) Relax all unstable sites, i.e., sites with $F_i \geq 1$: The force of an unstable site is reset to zero, $F_i \rightarrow 0$, and a fraction of it, αF_i, is distributed to each of its four nearest neighbors: $F_{nn} \rightarrow F_{nn} + \alpha F_i$. This step is repeated in a parallel update until there are no unstable sites left.

This two step rule is iterated indefinitely. The sequence of toppling events [step (2)] between application of the uniform drive [step (1)] defines an avalanche or earthquake. Since only a fraction 4α of the force is redistributed in each toppling, the model is nonconservative for $\alpha < 1/4$.

To completely define the model we need to specify the boundary conditions, by defining the dissipation parameter, α, at the sites on the boundaries (α_{bc}). As in OFC, we consider both ''free'' and ''open'' boundary conditions. The sites on the boundaries of the system can be considered to be bounded by fictitious sites with $F_i = -\infty$, which can never discharge and only absorb force from the boundary sites. In the case of open boundary conditions, the sites at the boundary have the same α parameter as all other sites in the bulk ($\alpha_{bc} = \alpha$). In the case of free boundary conditions, the sites at the boundary have their α parameters modified in order to have the same level of dissipation per reset event as for sites in the bulk. This latter condition implies $\alpha_{bc} = \alpha/(1-\alpha)$, except at corner sites where $\alpha_{bc,c} = \alpha/(1-2\alpha)$. The model with periodic boundary conditions is not critical [21–23], and will not be discussed. It is probably worthwhile to underline at this point that the model is completely deterministic, the only possible source of randomness coming from the initial conditions.

After a transient period of many earthquakes, the model settles into a statistically stationary state. One way to characterize this state is to measure statistical properties of the earthquakes. The size of an earthquake, s, is defined as the number of resets of the local force $F_i \rightarrow 0$ in the system in between applications of the uniform force. One can also measure the temporal duration t in terms of the parallel update, or the radius of gyration r of the sites which participated in the earthquake event.
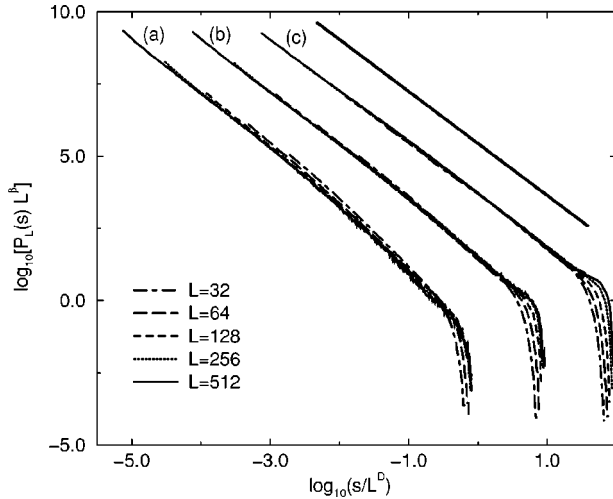
FIG. 1. Finite-size scaling plots of $P_L(s)$ in systems with open boundary conditions for (a) $\alpha = 0.15$, (b) $\alpha = 0.18$, and (c) $\alpha = 0.21$. The critical exponents are $D = 2$ and $\beta = 3.6$, and the slope of the straight line is $\tau = 1.8$. Statistics are derived from at least 10^9 avalanches per data set. For visual clarity, curves (b) and (c) have been shifted along the horizontal axis, $x \rightarrow x + 1$ and $x \rightarrow x + 2$, respectively.

Finite-size scaling

We focus on the probability distribution of earthquake sizes, s, in a system of size L, $P_L(s)$. If the model is critical then this distribution will have no scale other than the physical extent L and the lattice constant, which is set to unity. One ansatz that can describe critical behavior is the FSS ansatz that was previously used by OFC,

$$P_L(s) \sim L^{-\beta} G\left(\frac{s}{L^D}\right), \tag{1}$$

where G is a suitable scaling function, and β and D are critical exponents describing the scaling of the distribution function. As shown in Fig. 1, the model does not exhibit FSS. In this figure we chose $D = 2$ as the largest possible allowed value. Still the cutoff in the "collapsed" probability distribution moves to the right as L increases. Nevertheless, for earthquake sizes smaller than the cutoff, this figure shows what appears to be a convergence to a well-defined power law, $P_L(s) \sim s^{-\tau}$, as L increases with the power law exponent $\tau = \beta/D \simeq 1.8$ for both $\alpha = 0.18$ and 0.21, and possibly also for $\alpha = 0.15$.

III. MULTISCALING ANALYSIS

The FSS ansatz is only one possible description of critical behavior. As pointed out some time ago by Kadanoff *et al.*, some SOC phenomena are better described by a multifractal ansatz, rather than FSS [29]. This form was recently used to clarify the behavior of the Bak-Tang-Weisenfeld [1] model by Stella and co-workers, who measured different moments associated with the distribution [30]. For the OFC model, it appears to us that a clearer picture can be obtained by simply examining the probability distribution directly.
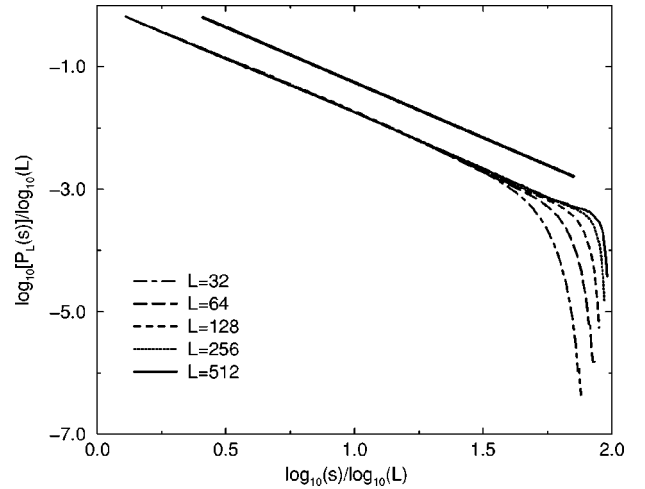

FIG. 2. Multiscaling plot of $P_L(s)$ for $\alpha = 0.21$ and open boundary conditions. The slope of the straight line is $\tau = 1.8$.

The multiscaling ansatz postulates for the probability distribution function $P_L(s)$ a form

$$\frac{\log P_L(s/s_o)}{\log(L/l_o)} = F\left(\frac{\log s/s_o}{\log L/l_o}\right), \tag{2}$$

where s_o and l_o are parameters which typically (but not always) reflect phenomena at small scales associated with the lattice [28]. Usually, a multiscaling analysis consists of choosing these two parameters in order to obtain the best collapse for different system sizes. This is quite different than FSS, where the critical exponents themselves, reflecting behavior at large scales, must be chosen in an *ad hoc* manner in order to obtain the "best" collapse. We do not attempt to collapse the data using the multiscaling form of Eq. (2). Instead, we define $l_o = s_o = 1$ to represent the smallest earthquake which occurs at only one site and involves only one discharge. Moreover, we define the dimension of an earthquake of size s in a systems of size L as

$$D_{av} = \log s / \log L,$$

and we denote with D_{max} the largest value of D_{av}.

In Fig. 2 we show the multiscaled probability distribution according to Eq. (2), for different system sizes for $\alpha = 0.21$. One observes immediately that all avalanches have dimension $D_{av} < 2$, as required, with the largest dimension D_{max} approaching 2 as the system size increases. Also it is clear that the shape of the cutoff function is sharpening as the system size increases. Since the largest dimension D_{max} cannot be larger than 2, we can infer from this that the cutoff region narrows to the region $D_{av} \rightarrow 2$, and becomes increasingly sharp.

Note that the increase of D_{max} as L increases is totally inconsistent with the notion that the OFC model is slightly off criticality, because in that case one would expect that the relative size of the largest dissipating events with respect to the maximum total force allowed in the system would decrease in larger systems. In fact what happens is exactly the opposite. In larger systems a larger fraction of the total en-
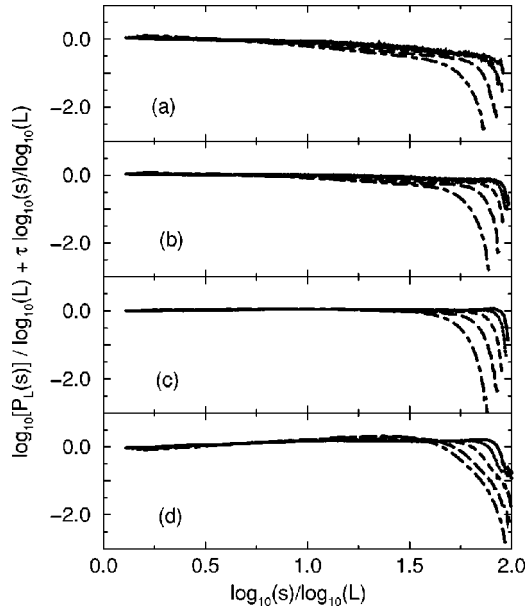
FIG. 3. Plots of F_{cutoff} for (a) $\alpha = 0.15$, (b) $\alpha = 0.18$, (c) $\alpha = 0.21$, and (d) $\alpha = 0.23$; we set $\tau = 1.8$. Boundary conditions are open. Different curves correspond, from left to right, to $L = 32, 64, 128, 256,$ and 512.
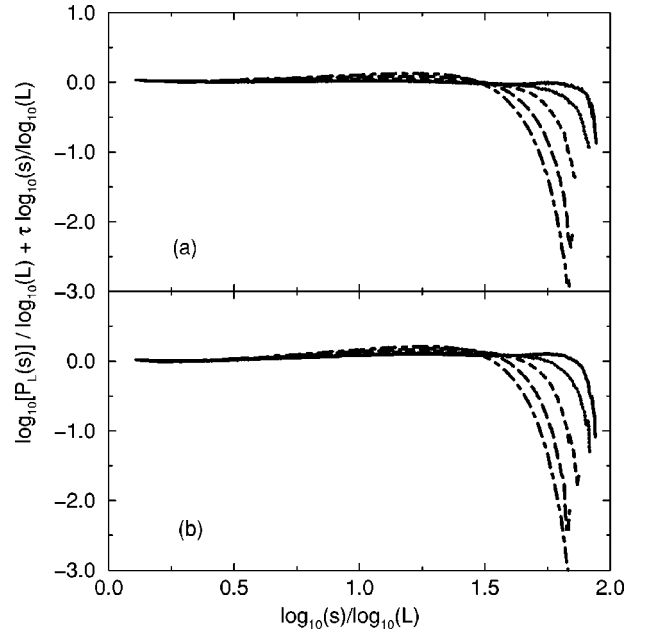

FIG. 4. Plots of F_{cutoff} in systems with free boundary conditions for (a) $\alpha = 0.17$ and (b) $\alpha = 0.20$. The exponent has been set to $\tau = 1.8$ as before. System sizes are $L = 32, 64, 128, 256,$ and 512.

ergy can be dissipated in the largest event that occurs, and D_{max} increases with L. This result is completely consistent with the nonconservative model being critical, rather than slightly off criticality.

In order to get more explicit visual information on the probability distribution, we try to subtract out the leading asymptotic term, which we propose is a power law, as suggested by Fig. 2. This is

$$\frac{\log P_L(s)}{\log(L)} = F(D_{av}) = -(\tau D_{av}) + F_{cutoff}(D_{av}), \quad (3)$$

where F_{cutoff} in the limit $L \to \infty$ should be constant up to a cutoff near D_{max}. We obtain a consistent picture for a range of α values by choosing $\tau = 1.8$, as shown in Fig. 3. Although it appears for small system sizes that smaller values of α give a steeper distribution with a larger value of τ (so the lines tend to decrease from left to right rather than remaining horizontal), it is clear from this figure that as the system size increases all the different values of α appear to reach the same value of $\tau = 1.8$, corresponding to a completely horizontal line in this figure. The deviation for small systems is more pronounced for $\alpha = 0.15$, and less again for $\alpha = 0.18$, becoming the smallest for $\alpha = 0.21$. For α closer to the conservative limit, as shown for $\alpha = 0.23$, the approach to the asymptotic horizontal behavior changes direction. That is, smaller systems appear to have a smaller exponent τ than larger systems, at least for small avalanches. Thus, rather than having the slope increase as L increases, for $\alpha > \sim 0.21$ the apparent slope decreases as L increases, as clearly demonstrated in this figure.

We ascribe the change of direction to a crossover effect of the conservative fixed point. For α close to 1/4, smaller ava-

lanches behave as avalanches would in the conservative system. It is only the larger avalanches that are affected by nonconservative dissipation. This is associated with the fact that as α approaches 1/4, each site can topple more and more times in a single earthquake. For any finite value of dissipation $(1 - 4\alpha)$, the maximum number of times that a site can reset in an earthquake is determined by this dissipation and is finite in the limit of large L. However for small systems, the largest avalanches are not large enough to be affected by dissipation, and the cutoff in the number of times that a site can topple is not determined by $(1 - 4\alpha)$ but by L. Then the cutoff in the number of topplings at a given site is the same as in a conservative system of the same size.

The results we have described thus far are for the model with open boundary conditions. Figure 4 shows that the same behavior occurs for the OFC model with free boundary conditions, with the same value $\tau = 1.8$. In this case, the asymptotic behavior as L increases is approached for decreasing apparent slope for both $\alpha = 0.17$ and 0.20, as is plainly evident in the figure. Again the cutoff appears to sharpen as L increases and approaches $D_{max} = 2$.

Our analysis indicates that the OFC model exhibits SOC with a universal power law distribution of events $P(s) \sim s^{-1.8}$ for $\log s < (D_{max} \log L)$. As shown in Fig. 3, the cutoff becomes sharper and sharper as L increases, with D_{max} approaching 2 from below as L increases for all values of α we have studied. As indicated above, the incorrect results obtained by OFC were due to the fact that there is a strong system size dependence that varies with α, and is not described by FSS. In addition to a systematic change in apparent τ as L increases, the largest dimension of earthquakes, D_{max}, increases only slowly toward 2, and the probability distribution of the avalanche dimensions becomes sharper at the cutoff. This means that large avalanches are suppressed

in small systems relative to the total amount of force in the system as compared to a larger system, and that a FSS ''fit'' for any L range will always give an apparent $D > 2$, which is not allowed. In this sense the model appears to violate FSS for all values of L.

IV. CONCLUSIONS

The main results of this paper are as follows: The non-conservative model on a two-dimensional lattice self-organizes into a critical state. The critical state is robust and universal over a range of values of the dissipation parameter α, and for different boundary conditions. The model does not exhibit finite-size scaling. The cutoff becomes sharper as L increases with largest earthquakes of dimension D_{max}, ap-proaching 2 from below. Nevertheless, the probability distribution of earthquake sizes is a power law with a universal exponent $\tau \simeq 1.8$. This value can be identified with an exponent $B = \tau - 1$ for the distribution of energy dissipated in earthquakes. According to the Gutenberg-Richter law this is a power law with $B = 0.8 - 1$, completely consistent with our result.

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